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Spacelike Weingarten Surfaces in R_1^3 and the Sine-Gordon Equation*

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In this paper we will establish a relation between spacelike Weingarten surfaces with condition $K + 2mH = m^2 + l^2$ in R_1^3 and solutions of the sine-Gordon equation. Moreover, we will give a method to construct new spacelike Weingarten surfaces from a given spacelike Weingarten surface in R_1^3 by Bäcklund transformation. © 1997 Academic Press

1. SPACELIKE WEINGARTEN SURFACES

Let R_1^3 be a 3-dimensional Minkowski space. We consider a spacelike Weingarten surface S in R_1^3 satisfying

$$K + 2mH = m^2 + l^2, \tag{1.1}$$

in which K is its Gauss curvature, H the mean curvature, and m, l are constants with $m^2 + l^2 > 0$.

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Suppose that S is a spacelike surface with condition (1.1) in R_1^3, and S has no umbilical point, so we can take the lines of curvature as its parametric curves, and write its fundamental forms as

$$I = A^2\,du^2 + B^2\,dv^2,$$
$$II = k_1 A^2\,du^2 + k_2 B^2\,dv^2,$$

where k_1, k_2 are the principal curvatures of S. Let

$$\begin{aligned} \omega^1 &= A\,du, & \omega^2 &= B\,dv, \\ \omega_1^3 &= k_1 A\,du, & \omega_2^3 &= k_2 B\,dv. \end{aligned} \tag{1.2}$$

Then the Gauss and Codazzi equations become

$$\begin{aligned} K &= -k_1 k_2, \\ (k_1 - k_2)A_v + (k_1)_v A &= 0, \\ (k_2 - k_1)B_u + (k_2)_u B &= 0. \end{aligned} \tag{1.3}$$

By use of the same method as in our previous paper [2], we can introduce new parameters locally on S, denoted still by u and v, such that its fundamental forms become

$$\begin{aligned} I &= \frac{1}{m^2 + l^2}\left(\cos^2\frac{\phi}{2}\cdot du^2 + \sin^2\frac{\phi}{2}\cdot dv^2\right), \\ II &= \frac{1}{\sqrt{m^2 + l^2}}\left(\sin\frac{\phi - \phi_0}{2}\cos\frac{\phi}{2}\cdot du^2 - \cos\frac{\phi - \phi_0}{2}\sin\frac{\phi}{2}\cdot dv^2\right), \end{aligned} \tag{1.4}$$

where

$$\sin\frac{\phi_0}{2} = -\frac{m}{\sqrt{m^2 + l^2}}, \qquad \cos\frac{\phi_0}{2} = \frac{l}{\sqrt{m^2 + l^2}}. \tag{1.5}$$

For convenience, the local parameters u, v of a spacelike Weingarten surface S with condition (1.1), for which its fundamental forms can be written by (1.4), are called *Tschebyscheff coordinates* on S, and ϕ is called the *Tschebyscheff angle*.

It follows from (1.4) that

$$\omega^1 = \frac{1}{\sqrt{m^2 + l^2}}\cos\frac{\phi}{2}\cdot du, \qquad \omega^2 = \frac{1}{\sqrt{m^2 + l^2}}\sin\frac{\phi}{2}\cdot dv, \tag{1.6}$$

$$\begin{aligned} \omega_1^3 = \omega_3^1 &= -k_1\omega^1 = -\sin\frac{\phi-\phi_0}{2}\cdot du, \\ \omega_2^3 = \omega_3^2 &= -k_2\omega^2 = \cos\frac{\phi-\phi_0}{2}\cdot dv, \end{aligned} \tag{1.7}$$

and

$$\omega_1^2 = -\omega_2^1 = \frac{1}{2}(\phi_v\, du + \phi_u\, dv). \tag{1.8}$$

The principal curvatures of S are

$$\begin{aligned} k_1 &= \sqrt{m^2+l^2}\cdot\frac{\sin((\phi-\phi_0)/2)}{\cos(\phi/2)}, \\ k_2 &= -\sqrt{m^2+l^2}\cdot\frac{\cos((\phi-\phi_0)/2)}{\sin(\phi/2)}. \end{aligned} \tag{1.9}$$

Obviously, the following Codazzi equations

$$d\omega_1^3 = \omega_1^2\wedge\omega_2^3, \qquad d\omega_2^3 = \omega_2^1\wedge\omega_1^3$$

hold automatically and the Gauss equation $d\omega_1^2 = K\omega^1\wedge\omega^2 = \omega_1^3\wedge\omega_3^2$ reads

$$\phi_{uu} - \phi_{vv} = -\sin(\phi-\phi_0). \tag{1.10}$$

We record these results as follows.

THEOREM 1.1. *Suppose S is a spacelike Weingarten surface without umbilics in R_1^3 satisfying condition* (1.1). *Then its Tschebyscheff angle ϕ is a solution of the sine-Gordon equation* (1.10).

Conversely, if $\tilde\phi$ is a solution of the sine-Gordon equation

$$\tilde\phi_{uu} - \tilde\phi_{vv} = -\sin\tilde\phi, \tag{1.11}$$

and m, l are constants such that $m^2+l^2>0$, then there exists locally a spacelike Weingarten surface S with condition (1.1) *in R_1^3 such that $\phi = \tilde\phi + \phi_0$ is its Tschebyscheff angle, where ϕ_0 is given by* (1.5).

2. DARBOUX LINE CONGRUENCE IN R_1^3

DEFINITION 2.1. Suppose S and S^* are two spacelike surfaces in R_1^3, and there exists a correspondence between them such that

(1) the distance between corresponding points is a constant λ, that is, $|\overrightarrow{PP^*}| = \lambda$, where $|v| = \sqrt{v \cdot v}$ for a spacelike vector v, and $|v| = \sqrt{-v \cdot v}$ for a timelike vector v;

(2) lines joining corresponding points are isoclinic with the surfaces S and S^*, that is, we can choose unit normal vectors e_3 and e_3^* on S and S^*, respectively, such that $t \cdot e_3 = -t \cdot e_3^* =$ constant, where t is the unit direction vector of $\overrightarrow{PP^*}$;

(3) the inner product of unit normal vectors at corresponding points of S and S^* is constant, i.e., $e_3 \cdot e_3^* =$ constant.

Then the congruence in R_1^3, formed by lines joining corresponding points of S and S^*, is called a Darboux line congruence associated with them.

Remark 2.1. Evidently, a pseudo-spherical line congruence between spacelike surfaces in R_1^3 (see [8]) is a special kind of Darboux line congruence (i.e., $t \cdot e_3 = 0 = t \cdot e_3^*$).

A Darboux line congruence is called spacelike or timelike according to its direction vector t being spacelike or timelike (the case that t is lightlike is not in our consideration).

If Σ is a timelike Darboux line congruence, then we can assume that

$$t \cdot e_3 = -t \cdot e_3^* = -\cosh \gamma = \text{const.}, \tag{2.1}$$

$$e_3 \cdot e_3^* = \cosh \theta = \text{const.}, \tag{2.2}$$

in this case, the end points of vectors $t, e_3, -e_3^*$ lie on the hyperbolic space

$$H^2 = \left\{ (x^1, x^2, x^3) \in R_1^3 : (x^1)^2 + (x^2)^2 - (x^3)^2 = -1,\, x^3 > 0 \right\},$$

where γ is the geodesic distance between t and e_3, and θ is the geodesic distance between e_3 and e_3^*. By the triangle inequality, we have

$$2\gamma \geq \theta > 0,$$

and

$$\coth \gamma \leq \coth \frac{\theta}{2},$$

i.e.,

$$\sinh^2\theta - (\cosh \theta - 1)^2 \coth^2\gamma \geq 0. \tag{2.3}$$

If Σ is a spacelike Darboux line congruence, we can assume that

$$t \cdot e_3 = -t \cdot e_3^* = -\sinh \gamma = \text{const.},$$

$$e_3 \cdot e_3^* = \cosh \theta = \text{const.},$$

then we always have

$$\sinh^2\theta - (\cosh\theta - 1)^2\tanh^2\gamma = \sinh^2\theta\left(1 - \tanh^2\frac{\theta}{2}\tanh^2\gamma\right) \geq 0. \quad (2.4)$$

THEOREM 2.1. *Suppose that S and S^* are two spacelike surfaces in R_1^3, and there exists a correspondence between them such that the lines joining corresponding points form a Darboux line congruence Σ in R_1^3. Then both S and S^* are spacelike Weingarten surfaces satisfying the same condition as* (1.1), *in which*

(1) *when Σ is a timelike Darboux line congruence,*

$$m = \frac{(\cosh\theta - 1)\cosh\gamma}{\lambda\sinh^2\gamma}, \qquad l^2 = \frac{\sinh^2\theta}{\lambda^2\sinh^2\gamma}\left(1 - \tanh^2\frac{\theta}{2}\cdot\coth^2\gamma\right), \tag{2.5}$$

(2) *when Σ is a spacelike Darboux line congruence,*

$$m = \frac{(\cosh\theta - 1)\sinh\gamma}{\lambda\cosh^2\gamma}, \qquad l^2 = \frac{\sinh^2\theta}{\lambda^2\cosh^2\gamma}\left(1 - \tanh^2\frac{\theta}{2}\cdot\tanh^2\gamma\right). \tag{2.6}$$

Proof. *Case* (1). Let Σ be a timelike Darboux line congruence associated with spacelike surfaces S and S^*. Then for a local orthonormal frame field $\{r; e_1, e_2, e_3\}$ on S, the direction vector t along congruence Σ can be expressed as

$$t = \sinh\gamma(\cos\psi e_1 + \sin\psi e_2) + \cosh\gamma e_3. \tag{2.7}$$

So the position vector r^* of S^* can be expressed as

$$r^* = r + \lambda t, \tag{2.8}$$

where $\lambda > 0$ is constant. We can choose the normal vector e_3^* of S^* so that

$$e_3\cdot e_3^* = \cosh\theta, \qquad t\cdot e_3 = -t\cdot e_3^* = -\cosh\gamma, \qquad e_3^*\cdot e_3^* = -1, \tag{2.9}$$

where both θ and γ are constant. Now, the normal vector e_3^* of S^* can be also written as

$$e_3^* = x_1e_1 + x_2e_2 - \cosh\theta e_3. \tag{2.10}$$

Then we get from (2.9)

$$\begin{aligned} x_1 &= -(\cosh\theta - 1)\coth\gamma\cos\psi \\ &\qquad + \sqrt{\sinh^2\theta - (\cosh\theta - 1)^2\coth^2\gamma}\cdot\sin\psi, \\ x_2 &= -(\cosh\theta - 1)\coth\gamma\sin\psi \\ &\qquad - \sqrt{\sinh^2\theta - (\cosh\theta - 1)^2\coth^2\gamma}\cdot\cos\psi. \end{aligned}$$

Put

$$\begin{aligned} \sin\frac{\phi_0}{2} &= -\frac{\cosh\theta - 1}{\sinh\theta}\coth\gamma, \\ \cos\frac{\phi_0}{2} &= \frac{\sqrt{\sinh^2\theta - (\cosh\theta - 1)^2\coth^2\gamma}}{\sinh\theta}. \end{aligned} \tag{2.11}$$

Then

$$x_1 = \sinh\theta\sin\left(\psi + \frac{\phi_0}{2}\right), \qquad x_2 = -\sinh\theta\cos\left(\psi + \frac{\phi_0}{2}\right). \tag{2.12}$$

Taking differentiation of (2.8) and using the structure equations we get

$$\begin{aligned} dr^* = {} & \left(\omega_1 + \lambda\cosh\gamma\omega_1^3\right)e_1 + \left(\omega^2 + \lambda\cosh\gamma\omega_2^3\right)e_2 \\ & + \lambda\sinh\gamma\left(d\psi + \omega_1^2\right)(-\sin\psi e_1 + \cos\psi e_2) \\ & + \lambda\sinh\gamma\left(\cos\psi\omega_1^3 + \sin\psi\omega_2^3\right)e_3. \end{aligned} \tag{2.13}$$

From $dr^* \cdot e_3^* = 0$ we have

$$\begin{aligned} & \lambda\sinh\gamma\sinh\theta\cos\frac{\phi_0}{2}\left(d\psi + \omega_1^2\right) \\ & \quad = \sinh\theta\left(\sin\left(\psi + \frac{\phi_0}{2}\right)\omega^1 - \cos\left(\psi + \frac{\phi_0}{2}\right)\omega^2\right) \\ & \qquad + \lambda\sinh\theta\cosh\gamma\left(\sin\left(\psi + \frac{\phi_0}{2}\right)\omega_1^3\right. \\ & \quad \left. - \cos\left(\psi + \frac{\phi_0}{2}\right)\omega_2^3\right) + \lambda\cosh\theta\sinh\gamma\left(\cos\psi\omega_1^3 + \sin\psi\omega_2^3\right). \end{aligned} \tag{2.14}$$

Taking differentiation of (2.14) and using the structure equation we get

$$
\begin{aligned}
&\lambda \sinh \gamma \sinh \theta \cos \frac{\phi_0}{2} \omega_1^3 \wedge \omega_2^3 \\
&\quad = \left(d\psi + \omega_1^2 \right) \wedge \left\{ \sinh \theta \left(\cos\left(\psi + \frac{\psi_0}{2} \right) \omega^1 + \sin\left(\phi + \frac{\phi_0}{2} \right) \omega^2 \right) \right. \\
&\qquad\qquad + \lambda \sinh \theta \cosh \gamma \left(\cos\left(\psi + \frac{\phi_0}{2} \right) \omega_1^3 + \sin\left(\psi + \frac{\phi_0}{2} \right) \omega_2^3 \right) \\
&\qquad\qquad \left. + \lambda \cosh \theta \sinh \gamma \left(-\sin \psi \omega_1^3 + \cos \psi \omega_2^3 \right) \right\}.
\end{aligned}
$$

Then by putting (2.14) in it we have

$$
\begin{aligned}
&\lambda^2 \sinh^2\gamma \sinh^2\theta \cos^2 \frac{\phi_0}{2} \omega_1^3 \wedge \omega_2^3 \\
&\quad = -\sinh^2\theta \omega^1 \wedge \omega^2 \\
&\qquad - \lambda \left(\sinh^2\theta \cosh \gamma + \sinh \theta \cosh \theta \sinh \gamma \sin \frac{\phi_0}{2} \right) \\
&\qquad\quad \times \left(\omega^1 \wedge \omega_2^3 + \omega_1^3 \wedge \omega^2 \right) \\
&\qquad - \lambda^2 \left(\sinh^2\theta \cosh^2\gamma + \cosh^2\theta \sinh^2\gamma \right. \\
&\qquad\qquad \left. + 2 \sinh \theta \cosh \theta \sinh \gamma \cosh \gamma \sin \frac{\phi_0}{2} \right) \omega_1^3 \wedge \omega_2^3. \qquad (2.15)
\end{aligned}
$$

We know that

$$
\begin{aligned}
\omega^1 \wedge \omega_2^3 + \omega_1^3 \wedge \omega^2 &= -(k_1 + k_2) \omega^1 \wedge \omega^2, \\
\omega_1^3 \wedge \omega_2^3 &= k_1 k_2 \cdot \omega^1 \wedge \omega^2.
\end{aligned}
\qquad (2.16)
$$

Then (2.14) can be reduced to

$$
k_1 k_2 - \frac{(\cosh \theta - 1) \cosh \gamma}{\lambda \sinh^2\gamma} (k_1 + k_2) + \frac{\sinh^2\theta}{\lambda^2 \sinh^2\gamma} = 0. \qquad (2.17)
$$

By comparing (2.17) with (1.1) we obtain (2.5).

Because (2.8) can be written as

$$
r = r^* + \lambda(-t),
$$

the same argument is valid for S^*. Then the surface S^* satisfies the same condition (2.17).

Case (2). Let Σ be a spacelike Darboux line congruence associated with spacelike surfaces S and S^*. Then for a local orthogonal frame field $\{r; e_1, e_2, e_3\}$ on S, the direction vector t along congruence Σ can be expressed as

$$t = \cosh \gamma(\cos \psi e_1 + \sin \psi e_2) + \sinh \gamma e_3. \tag{2.18}$$

By a similar procedure as in Case (1) we obtain

$$\begin{aligned}
&\lambda \cosh \gamma \sinh \theta \cos \frac{\phi_0}{2}\left(d\psi + \omega_1^2\right) \\
&\quad = \sinh \theta \left(\sin\left(\psi + \frac{\phi_0}{2} \right) \omega^1 - \cos\left(\psi + \frac{\phi_0}{2} \right) \omega^2 \right) \\
&\qquad + \lambda \sinh \theta \sinh \gamma \left(\sin\left(\psi + \frac{\phi_0}{2} \right) \omega_1^3 \right. \\
&\quad \left. - \cos\left(\psi + \frac{\phi_0}{2} \right) \omega_2^3 \right) + \lambda \cosh \theta \cosh \gamma \left(\cos \psi \omega_1^3 + \sin \psi \omega_2^3\right),
\end{aligned} \tag{2.19}$$

in which ϕ_0 is given by

$$\begin{aligned}
\sin \frac{\phi_0}{2} &= -\frac{\cosh \theta - 1}{\sinh \theta} \cdot \tanh \gamma, \\
\cos \frac{\phi_0}{2} &= \frac{\sqrt{\sinh^2\theta - (\cosh \theta - 1)^2 \tanh^2\gamma}}{\sinh \theta}.
\end{aligned} \tag{2.20}$$

A similar argument as above gives

$$k_1 k_2 - \frac{(\cosh \theta - 1) \sinh \gamma}{\lambda \cosh^2\gamma}(k_1 + k_2) + \frac{\sinh^2\theta}{\lambda^2 \cosh^2\gamma} = 0. \tag{2.21}$$

3. BÄCKLUND TRANSFORMATION

Suppose Σ is a timelike Darboux line congruence associated with spacelike surfaces S and S^* in R_1^3, so S and S^* are Weingarten spacelike surfaces with condition (1.1), and m, l are given by (2.5). Let u, v be the

Tschebyscheff coordinates on S. Then

$$\begin{aligned}
\omega^1 &= \frac{\lambda \sinh \gamma}{\sinh \theta} \cos \frac{\phi}{2} \cdot du, \qquad \omega^2 = \frac{\lambda \sinh \gamma}{\sinh \theta} \sin \frac{\phi}{2} \cdot dv, \\
\omega_1^3 &= -\sin \frac{\phi - \phi_0}{2} \cdot du, \qquad \omega_2^3 = \cos \frac{\phi - \phi_0}{2} \cdot dv, \\
\omega_1^2 &= \frac{1}{2} (\phi_v \, du + \phi_u \, dv),
\end{aligned} \tag{3.1}$$

where ϕ_0 is given by (2.11). Let

$$\psi = -\frac{\phi^*}{2}, \tag{3.2}$$

and put (3.1) into (2.14). Then we have

$$\begin{aligned}
&\sinh \theta \left(d\psi + \omega_1^2 \right) \\
&\quad = -du \cdot \left(\sin \frac{\phi^* - \phi_0}{2} \cos \frac{\phi - \phi_0}{2} \right. \\
&\qquad\qquad \left. + \cosh \theta \cos \frac{\phi^* - \phi_0}{2} \sin \frac{\phi - \phi_0}{2} \right) \\
&\qquad - dv \cdot \left(\cos \frac{\phi^* - \phi_0}{2} \sin \frac{\phi - \phi_0}{2} \right. \\
&\qquad\qquad \left. + \cosh \theta \sin \frac{\phi^* - \phi_0}{2} \cos \frac{\phi - \phi_0}{2} \right),
\end{aligned} \tag{3.3}$$

that is,

$$\begin{aligned}
\sinh \theta \cdot \frac{\phi_u^* - \phi_v}{2} &= \sin \frac{\phi^* - \phi_0}{2} \cos \frac{\phi - \phi_0}{2} \\
&\quad + \cosh \theta \cos \frac{\phi^* - \phi_0}{2} \sin \frac{\phi - \phi_0}{2}, \\
\sinh \theta \cdot \frac{\phi_v^* - \phi_u}{2} &= \cos \frac{\phi^* - \phi_0}{2} \sin \frac{\phi - \phi_0}{2} \\
&\quad + \cosh \theta \sin \frac{\phi^* - \phi_0}{2} \cos \frac{\phi - \phi_0}{2}.
\end{aligned} \tag{3.4}$$

By verifying directly we get the following

PROPOSITION 3.1. *If ϕ is a solution of the sine-Gordon equation*

$$\phi_{uu} - \phi_{vv} = -\sin(\phi - \phi_0), \tag{3.5}$$

then, for any $\theta \neq 0$, Eqs. (3.4) *are completely integrable for ϕ^*, and its solution ϕ^* satisfies the same Eq.* (3.5).

Therefore (3.4) give a Bäcklund transformation of solutions of the sine-Gordon equation (3.5). Furthermore, if Σ is a spacelike Darboux line congruence associated with spacelike surfaces S and S^* in R_1^3, then we obtain the same Bäcklund transformation (3.4) from (2.19).

In fact, we can prove the following theorem.

THEOREM 3.1. *Suppose Σ is a spacelike* (*or timelike*) *Darboux line congruence associated with spacelike surfaces in R_1^3. Then both S and S^* are spacelike Weingarten surfaces satisfying the same condition as* (1.1), *and their Tschebyscheff angles ϕ and ϕ^* are both solutions of the sine-Gordon equation* (3.5), *and they are related to each other by a Bäcklund transformation* (3.4), *where ϕ_0 is given by* (1.5).

Proof. We will prove this theorem only for Σ being the timelike case, and the proof in another case is similar.

Recalling the definition of cross product of vectors in R_1^3 (see [1]), we have

$$\begin{aligned} e_3 \times e_3^* &= x^1 \cdot e_3 \times e_1 + x^2 \cdot e_3 \times e_2 = x^2 e_1 - x^1 e_2 \\ &= -\sinh\theta\left(\cos\left(\psi + \frac{\phi_0}{2}\right)e_1 + \sin\left(\psi + \frac{\phi_0}{2}\right)e_2\right) \\ &= -\sinh\theta\left(\cos\frac{\phi_0}{2}\cdot\tau + \sin\frac{\phi_0}{2}\cdot\tau^{\perp}\right), \end{aligned} \tag{3.6}$$

where

$$\tau = \cos\psi\cdot e_1 + \sin\psi\cdot e_2 \tag{3.7}$$

is the unit vector of orthogonal projection of the direction vector t of Σ on tangent space of S, and

$$\tau^{\perp} = -\sin\psi\cdot e_1 + \cos\psi\cdot e_2.$$

Then the vector

$$e = \cos\frac{\phi_0}{2}\cdot\tau + \sin\frac{\phi_0}{2}\cdot\tau^{\perp} = \cos\left(\psi + \frac{\phi_0}{2}\right)e_1 + \sin\left(\psi + \frac{\phi_0}{2}\right)e_2 \quad (3.8)$$

is tangent to both S and S^*, and $\phi_0/2$ is the angle between τ and the vector e.

As stated at the beginning of this section we have a local frame field $\{r; e_1, e_2, e_3\}$, in which e_1, e_2 are along the principle directions of S. In addition we have two other local frame fields $\{r; \tau, \tau^{\perp}, e_3\}$ and $\{r; e, e^{\perp}, e_3\}$ on S, where

$$\begin{aligned} e^{\perp} = e \times e_3 &= -\sin\frac{\phi_0}{2}\cdot\tau + \cos\frac{\phi_0}{2}\cdot\tau^{\perp} \\ &= -\sin\left(\psi + \frac{\phi_0}{2}\right)e_1 + \cos\left(\psi + \frac{\phi_0}{2}\right)e_2, \end{aligned} \quad (3.9)$$

and a local frame field $\{r^*; e^*, e^{*\perp}, e_3^*\}$ on S^*, where

$$\begin{aligned} e^* &= -e = -\cos\left(\psi^* - \frac{\phi_0}{2}\right)\cdot e_1 - \sin\left(\psi^* - \frac{\phi_0}{2}\right)\cdot e_2 \\ &= -\cos\frac{\phi^* - \phi_0}{2}\cdot e_1 + \sin\frac{\phi^* - \phi_0}{2}\cdot e_2, \\ e^{*\perp} &= e^* \times e_3^* \\ &= \cosh\theta\left(-\sin\left(\psi + \frac{\phi_0}{2}\right)\cdot e_1 + \cos\left(\psi + \frac{\phi_0}{2}\right)\cdot e_2\right) + \sinh\theta\cdot e_3 \\ &= \cosh\theta\left(\sin\frac{\phi^* - \phi_0}{2}\cdot e_1 + \cos\frac{\phi^* - \phi_0}{2}\cdot e_2\right) + \sinh\theta\cdot e_3. \end{aligned} \quad (3.10)$$

Putting (3.3) into (2.13), we get

$$\begin{aligned} dr^* = &\left(\frac{\lambda\sinh\gamma}{\sinh\theta}\cos\frac{\phi}{2} - \lambda\cosh\gamma\sin\frac{\phi - \phi_0}{2}\right)du\cdot e_1 \\ &+ \left(\frac{\lambda\sinh\gamma}{\sinh\theta}\sin\frac{\phi}{2} + \lambda\cosh\gamma\cos\frac{\phi - \phi_0}{2}\right)dv\cdot e_2 \\ &- \frac{\lambda\sinh\gamma}{\sinh\theta}\left\{\left(\sin\frac{\phi^* - \phi_0}{2}\cos\frac{\phi - \phi_0}{2}\right.\right. \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \cosh\theta \cos\frac{\phi^* - \phi_0}{2} \sin\frac{\phi - \phi_0}{2}\Big)\, du \\
&\qquad\quad + \Big(\cos\frac{\phi^* - \phi_0}{2} \sin\frac{\phi - \phi_0}{2} \\
&\qquad\qquad\quad + \cosh\theta \sin\frac{\phi^* - \phi_0}{2} \cos\frac{\phi - \phi_0}{2}\Big)\, dv\Big\} \cdot \tau^{\perp} \\
&- \lambda \sinh\gamma \Big(\cos\frac{\phi^*}{2} \sin\frac{\phi - \phi_0}{2}\, du + \sin\frac{\phi^*}{2} \cos\frac{\phi - \phi_0}{2}\, dv\Big) \cdot e_3.
\end{aligned}
\tag{3.11}
$$

Denote $\{\xi^{*1}, \xi^{*2}\}$ the coframe dual to $\{r^*; e^*, e^{*\perp}, e_3^*\}$ on S^*. So

$$
\begin{aligned}
\xi^{*1} &= dr^* \cdot e^* \\
&= -\frac{\lambda \sinh\gamma}{\sinh\theta}\Big(\cos\frac{\phi^*}{2} \cos\frac{\phi - \phi_0}{2}\, du - \sin\frac{\phi^*}{2} \sin\frac{\phi - \phi_0}{2}\, dv\Big), \\
\xi^{*2} &= dr^* \cdot e^{*\perp} \\
&= -\frac{\lambda \sinh\gamma}{\sinh\theta}\Big(\cos\frac{\phi^*}{2} \sin\frac{\phi - \phi_0}{2}\, du + \sin\frac{\phi^*}{2} \cos\frac{\phi - \phi_0}{2}\, dv\Big).
\end{aligned}
\tag{3.12}
$$

To find out ξ_1^{*3} and ξ_2^{*3} we take differentiation of (2.10), and get

$$
\begin{aligned}
de_3^* &= \left(-x^2 e_1 + x^1 e_2\right)\left(d\psi + \omega_1^2\right) + x^3\left(\omega_1^3 e_1 + \omega_2^3 e_2\right) + \left(x^1 \omega_1^3 + x^2 \omega_2^3\right) e_3 \\
&= -\Big\{\Big(\sin\frac{\phi^* - \phi_0}{2} \cos\frac{\phi - \phi_0}{2} + \cosh\theta \cos\frac{\phi^* - \phi_0}{2} \sin\frac{\phi - \phi_0}{2}\Big)\, du \\
&\quad + \Big(\cos\frac{\phi^* - \phi_0}{2} \sin\frac{\phi - \phi_0}{2} + \cosh\theta \sin\frac{\phi^* - \phi_0}{2} \cos\frac{\phi - \phi_0}{2}\Big)\, dv\Big\} e \\
&\quad + \cosh\theta \Big(\sin\frac{\phi - \phi_0}{2}\, du \cdot e_1 - \cos\frac{\phi - \phi_0}{2}\, dv \cdot e_2\Big) \\
&\quad + \sinh\theta \Big(\sin\frac{\phi^* - \phi_0}{2} \sin\frac{\phi - \phi_0}{2}\, du \\
&\qquad\qquad - \cos\frac{\phi^* - \phi_0}{2} \cos\frac{\phi - \phi_0}{2}\, dv\Big) e_3,
\end{aligned}
\tag{3.13}
$$

there we have

$$\begin{aligned} \xi_1^{*3} &= de_3^* \cdot e^* \\ &= \sin\frac{\phi^* - \phi_0}{2}\cos\frac{\phi - \phi_0}{2}\,du + \cos\frac{\phi^* - \phi_0}{2}\sin\frac{\phi - \phi_0}{2}\,dv, \\ \xi_2^{*3} &= de_3^* \cdot e^{*\perp} \\ &= \sin\frac{\phi^* - \phi_0}{2}\sin\frac{\phi - \phi_0}{2}\,du - \cos\frac{\phi^* - \phi_0}{2}\cos\frac{\phi - \phi_0}{2}\,dv. \end{aligned} \tag{3.14}$$

Consider a local frame field $\{r^*; e_1^*, e_2^*, e_3^*\}$ on S^* such that

$$\begin{aligned} e^* &= \cos\frac{\phi - \phi_0}{2}e_1^* - \sin\frac{\phi - \phi_0}{2}e_2^*, \\ e^{*\perp} &= \sin\frac{\phi - \phi_0}{2}e_1^* + \cos\frac{\phi - \phi_0}{2}e_2^*, \end{aligned} \tag{3.15}$$

and denote its dual as $\{\omega^{*1}, \omega^{*2}\}$. Then we have

$$\begin{aligned} \omega^{*1} &= \cos\frac{\phi - \phi_0}{2}\xi^{*1} + \sin\frac{\phi - \phi_0}{2}\xi^{*2} = -\frac{\lambda\sinh\gamma}{\sinh\theta}\cos\frac{\phi^*}{2}\,du, \\ \omega^{*2} &= -\sin\frac{\phi - \phi_0}{2}\xi^{*1} + \cos\frac{\phi - \phi_0}{2}\xi^{*2} = -\frac{\lambda\sinh\gamma}{\sinh\theta}\sin\frac{\phi^*}{2}\,dv, \\ \omega_1^{*3} &= \cos\frac{\phi - \phi_0}{2}\xi_1^{*3} + \sin\frac{\phi - \phi_0}{2}\xi_2^{*3} = \sin\frac{\phi^* - \phi_0}{2}\,du, \\ \omega_2^{*3} &= -\sin\frac{\phi - \phi_0}{2}\xi_1^{*3} + \cos\frac{\phi - \phi_0}{2}\xi_2^{*3} = -\cos\frac{\phi^* - \phi_0}{2}\,dv. \end{aligned} \tag{3.16}$$

Comparing the above formulas with (3.1), we know that u and v are Tschebyscheff coordinates on S^*, and ϕ^* is its Tschebyscheff angle.

Remark 3.1. Theorem 3.1 implies a way to construct new Weingarten spacelike surfaces with condition (1.1) in R_1^3 from a given one. Without loss of generality, we can always assume $m \geq 0$ in (1.1) (if necessary, change the orientation of S).

THEOREM 3.2. *Suppose S is a spacelike surface with condition* (1.1) *in R_1^3, $m \geq 0$. For any given real number $\theta > 0$ and* $\tanh(\theta/2) \neq m/\sqrt{m^2 + l^2}$, *we can construct a Darboux line congruence Σ such that the solution of the completely integrable equation* (3.4) *is the Tschebyscheff angle of the corresponding surface S^*.*

Proof. Assume that $\tanh(\theta/2) < m/\sqrt{m^2+l^2}$. Then we can choose γ such that

$$\coth\gamma = \coth\frac{\theta}{2}\cdot\frac{m}{\sqrt{m^2+l^2}} > 1.$$

Let (u, v) be the Tschebyscheff coordinates on S. Then Eq. (3.4) is completely integrable by Theorem 3.1 and Proposition 3.1. Let ϕ^* be the solution of (3.4) such that $\phi^*(u_0, v_0) = \phi(u_0, v_0)$. Put

$$\lambda = \frac{1}{\sqrt{m^2+l^2}}\cdot\frac{\sinh\theta}{\sinh\gamma},$$

then (2.5), (2.11), (3.1) hold. Let

$$t = \sinh\gamma\left(\cos\frac{\phi^*}{2}\cdot e_1 - \sin\frac{\phi^*}{2}\cdot e_2\right) + \cosh\gamma e_3, \tag{3.17}$$

and

$$r^* = r + \lambda t. \tag{3.18}$$

We want to prove that r^* is a spacelike surface, and that the above formula gives a timelike Darboux line congruence in R_1^3 associated with S and S^*.

Let

$$e_3^* = -\sinh\theta\left(\sin\frac{\phi^*-\phi_0}{2}e_1 + \cos\frac{\phi^*-\phi_0}{2}e_2\right) - \cosh\theta e_3. \tag{3.19}$$

By differentiation of (3.18) we get

$$\begin{aligned} dr^* = {}& \omega^1 e_1 + \omega^2 e_2 - \lambda\sinh\gamma\left(\sin\frac{\phi^*}{2}e_1 + \cos\frac{\phi^*}{2}e_2\right) \\ & \times\left(\frac{\phi_u^*-\phi_v}{2}\,du + \frac{\phi_v^*-\phi_u}{2}\,dv\right) \\ & + \lambda\cosh\gamma\left(\omega_1^3 e_1 + \omega_2^3 e_2\right) \\ & + \lambda\sinh\gamma\left(\cos\frac{\phi^*}{2}\,\omega_1^3 - \sin\frac{\phi^*}{2}\,\omega_2^3\right)e_3. \end{aligned} \tag{3.20}$$

By use of (3.1) and (3.4) we obtain

$$
\begin{aligned}
dr^* \cdot e_3^* = & -\sinh\theta\left(\sin\frac{\phi^* - \phi_0}{2}\omega^1 + \cos\frac{\phi^* - \phi_0}{2}\omega^2\right) \\
& + \lambda\sinh\theta\sinh\gamma\cos\frac{\phi_0}{2}\left(\frac{\phi_u^* - \phi_v}{2}du + \frac{\phi_v^* - \phi_u}{2}dv\right) \\
& - \lambda\sinh\theta\cosh\gamma\left(\sin\frac{\phi^* - \phi_0}{2}\omega_1^3 + \cos\frac{\phi^* - \phi_0}{2}\omega_2^3\right) \\
& + \lambda\cosh\theta\sinh\gamma\left(\cos\frac{\phi^*}{2}\omega_1^3 - \sin\frac{\phi^*}{2}\omega_2^3\right) \\
= & \ 0,
\end{aligned}
\tag{3.21}
$$

thus e_3^* is a normal vector of S^*. From (3.19), $e_3^* \cdot e_3^* = -1$. Then S^* is a spacelike surface.

From the definition of t and e_3^*, we have

$$
\begin{aligned}
t \cdot e_3 = -t \cdot e_3^* = -\cosh\gamma = \text{const.}, \\
e_3 \cdot e_3^* = \cosh\gamma = \text{const.}, \qquad t \cdot t = -1,
\end{aligned}
\tag{3.22}
$$

hence the line congruence given by (3.18) is a timelike Darboux line congruence. By Theorem 3.1, ϕ^* is the Tschebyscheff angle of S^*.

If $\tanh(\theta/2) > m/\sqrt{m^2 + l^2}$, then we can choose γ such that

$$
\tanh\gamma = \coth\frac{\theta}{2} \cdot \frac{m}{\sqrt{m^2 + l^2}} < 1, \tag{3.23}
$$

and let

$$
t = \cosh\gamma\left(\cos\frac{\phi^*}{2} \cdot e_1 - \sin\frac{\phi^*}{2} \cdot e_2\right) + \sinh\gamma e_3, \tag{3.24}
$$

$$
r^* = r + \lambda t. \tag{3.25}
$$

The remainder of proof is similar to above; we omit the details here.

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